# §1. Real numbers

## §1.1 Introduction

most basic idea: counting!

$1, 2, 3, 4, 5, \ldots$

→ denote by letter $\mathbb{N}$:

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

special numbers: prime numbers

$$15 = 3 \cdot 5$$

$$13500 = 3^3 \cdot 2^2 \cdot 5^3$$

(3, 2 and 5 are prime numbers, only divisible by themselves and 1)

Suppose now we want to solve the simple equation $x + 8 = 4$

one reaction: has no answer

alternative: postulate "$-4$" to be the solution

→ negetive numbers

Altogether, we obtain the integers:

$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

**Note:** the number 0 is defined as the solution to the equation $x+4=4$

Consider now the equation:

$$3x + 2 = 4$$

no integer is a solution!

$\rightarrow$ solution leads to fractional or "rational" numbers:

$$\mathbb{Q} = \left\{ \begin{array}{l} \text{all numbers of the form } \frac{p}{q}, \\ \text{where } p \text{ and } q \text{ are integers and } q \neq 0 \end{array} \right\}$$

**Notation:** $\frac{1}{2} = 0.5$ (decimal fractions)

for example $27 + \frac{5}{10} + \frac{3}{100} + \frac{2}{1000} + \frac{8}{10000}$

$$= 27.5328$$

Let us now move on. Consider

$$x^2 = 2$$

observe $1^2 = 1$, $2^2 = 4$

$\rightarrow$ $x$ should be between 1 and 2

take for example $x = 1.5$, then

$$(1.5)^2 = 2.25 > 2$$
$$(1.4)^2 = 1.96 < 2$$

→ obtain a sequence

$1, 2, \frac{3}{2}, \frac{7}{5}, \frac{141}{100}, \frac{71}{50}, \frac{707}{500}, \ldots$

At some point in history it was realized:

<u>no</u> rational number solves the equation $x^2 = 2$ !

<u>Very important</u>, will give a proof later

For now: postulate the solution to be $\sqrt{2}$

"irrational number"  "root" of 2

other irrational numbers: $\sqrt{97}$,

$$\sqrt[5]{2\sqrt{97}} - \frac{5}{3} + \sqrt[3]{2 + \sqrt{52}}$$

<u>Note:</u>

i) Not every root is irrational!

for example, $\sqrt{4} = 2$, $\sqrt{\frac{9}{25}} = \frac{3}{5}$

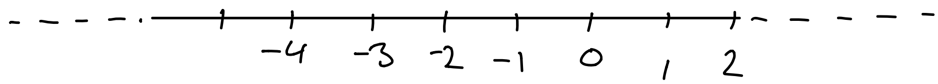
need to check in each case!

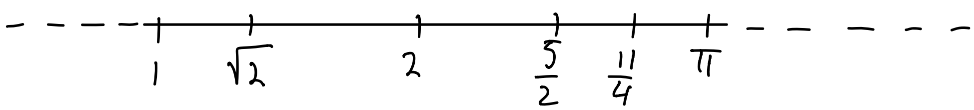
ii) Not all irrational numbers arise as roots of rationals or combinations thereof!

famous example: $\pi$

The collection of all integers, rationals, and irrationals are called "real numbers" and denoted by $\mathbb{R}$.

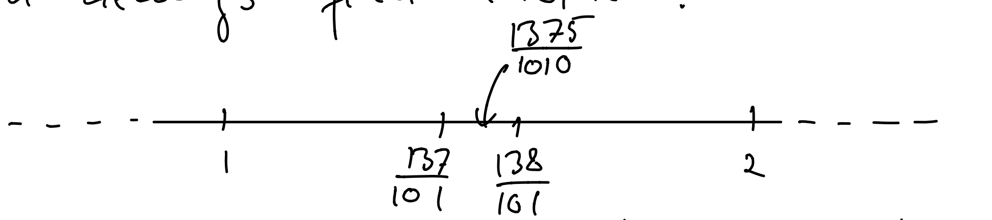
geometric visualization:




rational and irrational numbers

Between any two rational numbers, you can always find another:




In fact, there are infinitely many!

We say: the rational numbers are "densely" spread along the line

Where are the the irrational numbers?

"dense" ⟶ "continuous"

Will define carefully what the difference is later on.

Let us now move on. Consider the equation

$$x^2 + 1 = 0$$

→ need to find a number whose square is $-1$ !

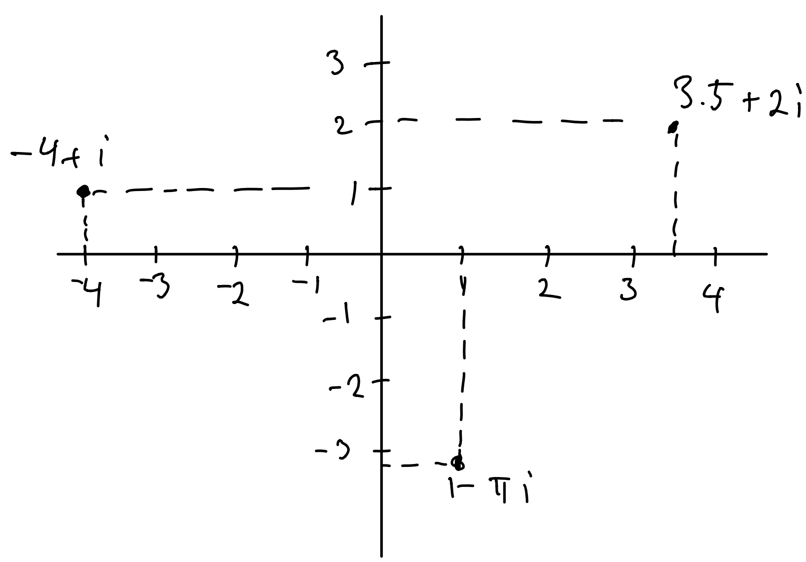
Sounds impossible !

→ postulate the number $i$ (imaginary) such that $i^2 = -1$

Have arrived at the "complex numbers":

$$\underbrace{a}_{\text{real}} + i\underbrace{b}_{\text{numbers}}$$

→ denoted by $\mathbb{C}$ :



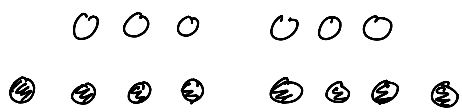

Complex numbers are "algebraically complete": any "polynomial equation", such as

$$x^5 - 5x^4 + 30x^3 - 50x^2 + 55x - 21 = 0,$$

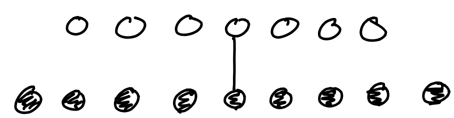
can be solved with complex numbers !

## Some historical comments:

Ancient Greeks represented numbers with help of pebbles:



even numbers
(two identical rows)

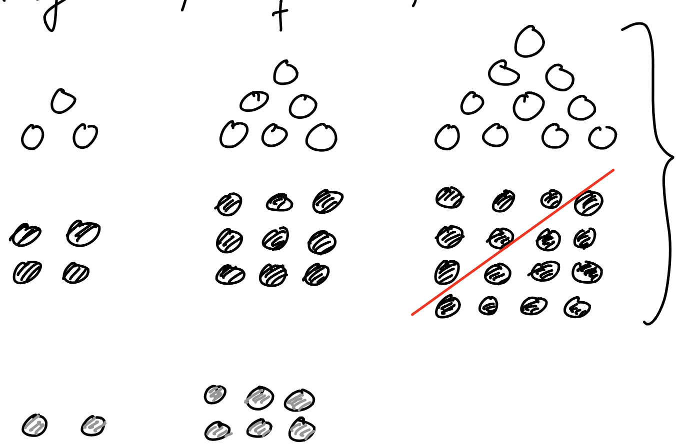


odd numbers
(if arranged in two identical rows, always leaves a separate pebble)

addition is done by regrouping the pebbles

$\to$ sum of even numbers is even
sum of an even number of odd numbers is even

Triangular, square, and oblong numbers:

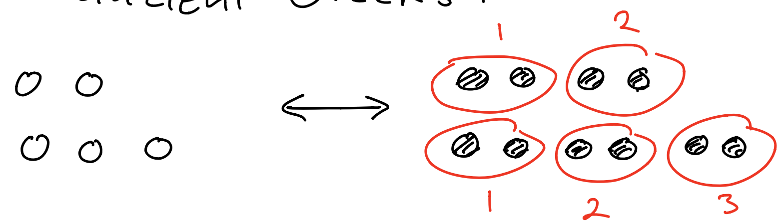


$\to$ square numbers are composed of triangular numbers ratios and proportions:

the pairs 2,3 and 4, 6 are in proportion

$\longrightarrow$ modern statement: $\frac{2}{3} = \frac{4}{6}$

But for ancient Greeks:



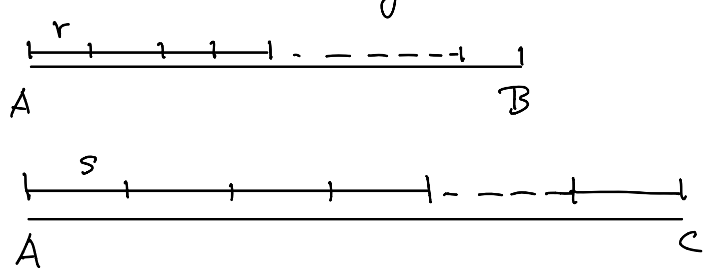

A ratio was for them <u>not</u> a number, but a way to <u>compare</u> numbers.

(2 to 3 is like 4 to 6)

There was no concept of adding or subtracting them like we do with modern ratios

Incommensurability:

Consider the segments AB and AC:



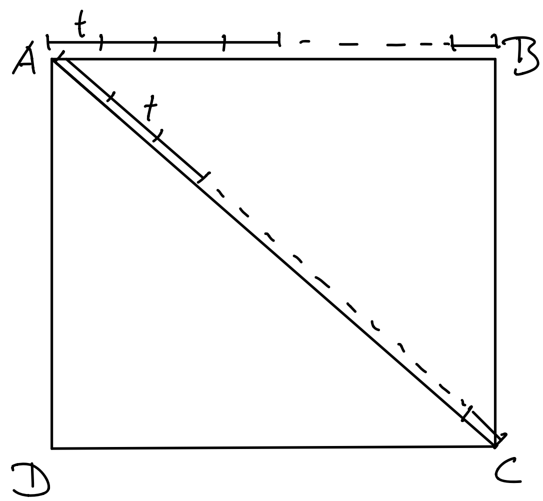

AB is "measured" in terms of r, (units of
AC is "measured" in terms of s. measurement)

Now let's imagine we want to measure

with a <u>common</u> unit $t$.

$\longrightarrow$ Two lengths are "commensurable" if they can be measured with the same unit $t$.

Next, let's imagine we want to measure sides and the diagonal of a square with the same unit:




assign $|AB| = 1$

then $|AC| = \sqrt{2}$

$\longrightarrow$ $AB$ and $AC$ are "incommensurable"!

Otherwise: $|AB| = t \cdot n$, $|AC| = t \cdot m$, $n, m \in \mathbb{N}$

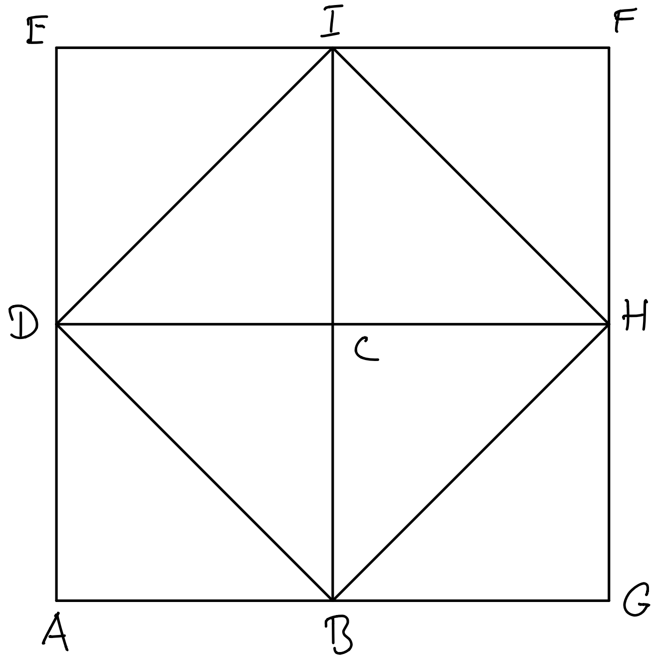
$$\text{and} \quad \frac{|AB|}{|AC|} = \frac{t \cdot n}{t \cdot m} = \frac{n}{m} \in \mathbb{Q}$$

But we know $\sqrt{2} \notin \mathbb{Q}$!

But how do we know this?

Was discovered by Pythagoreans of ancient Greece!

Let's have a look at their proof:




Assume now that the segments DH and DB are commensurable, i.e.

$|DH| = m \cdot t$ and $n$ and $m$ co-prime (have no common factors)

$|DB| = n \cdot t$

Then DBHI and AGFE represent square numbers, i.e. $\text{area}(DBHI) = n^2$, $\text{area}(AGFE) = m^2$, and in addition $m^2 = 2n^2$

$\rightarrow m^2$ is even $\rightarrow m$ is even

$\rightarrow m^2$ can be divided into four

$\rightarrow \text{area}(ABCD) = k$ (where $k \cdot 4 = m^2$)

But then $\text{area}(DBHI) = 2 \cdot \text{area}(ABCD)$

Hence DBHI represents a square number that is even $\longrightarrow$ n is even

$\lightning$ contradiction (n and m were co-prime!)

Hence n and m are incommensurable! $\square$

## §1.2 Axiomatic approach

assume fundamental laws (axioms)

$\longrightarrow$ derive everything else

3 classes of axioms:

A) The field axioms (describe laws like $+, \cdot, -, \div$)

B) Ordering axioms (describe $<, \leq, >, \geq$)

C) Completeness axiom (describes difference between $\mathbb{Q}$ and $\mathbb{R}$)

A) Field axioms

$\mathbb{R}$ is a "set". On this set there are two operations:

1) : $(\underset{\mathbb{R}}{a}, \underset{\mathbb{R}}{b}) \longmapsto (a+b) \in \mathbb{R}$

2) : $(a, b) \longmapsto (a \cdot b) \in \mathbb{R}$

satisfying the following axioms

| | $+$ | $\cdot$ |
|---|---|---|
| Commutative law | $a+b = b+a$ | $a \cdot b = b \cdot a$ |
| Associative law | $(a+b)+c = a+(b+c)$ | $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ |
| neutral element | $\exists\ 0 \in \mathbb{R}$ with $a + 0 = a$ | $\exists\ 1 \in \mathbb{R}$ with $a \cdot 1 = a$ |
| | $0 \neq 1$ | |
| Inverse element | $\forall a\ \exists b$ with $a+b = 0$ | $\forall\ a \neq 0\ \exists b$ with $a \cdot b = 1$ |
| Distributive law | $a \cdot (b+c) = a \cdot b + a \cdot c$ | |

Remark:

i) A set satisfying these axioms is called a field. $\mathbb{R}$ is an example of a field.

There are other fields, e.g.

$$K = \{0, 1\}$$

with the operations

| | |
|---|---|
| $0+0=0$ | $0 \cdot 0 = 0$ |
| $0+1=1$ | $0 \cdot 1 = 0$ |
| $1+0=1$ | $1 \cdot 0 = 0$ |
| $1+1=0$ | $1 \cdot 1 = 1$ |

ii) For $a$ there exists <u>exactly</u> one $b$ with $a+b=0$. Suppose there is $b'$ with $a+b'=0$, then

$$
\begin{aligned}
b' &= b' + 0 && \text{neutr. } + \\
b' &= b' + (a+b) && \\
b' &= (b'+a)+b && \text{Assoc. } + \\
b' &= (a+b')+b && \text{Comm. } + \\
b' &= 0 + b && \\
b' &= b + 0 && \text{Comm. } + \\
b' &= b && \text{neutr. } +
\end{aligned}
$$

This element is denoted by $-a$:

$$a + (-b) = a - b$$

$\longrightarrow$ defines subtraction and difference

iii) For each $a \neq 0$ $\exists$ $b$ with $a \cdot b = 1$

(analogous to ii))

notation $a^{-1}$ or $\frac{1}{a}$

$a \cdot (b^{-1}) = \frac{a}{b}$ $\longrightarrow$ defines quotient and division